Kupuli: A Multilingual and Multidisciplinary Journal
Published by Government General Degree College, Manbazar-II, Purulia

ISSN: Nil Volume – 1 Issue- January- June 2023

A NOTE ON RATES OF $A^{\mathcal{I}}$-STATISTICAL CONVERGENCE OF OPERATORS IN THE SPACE OF LOCALLY INTEGRABLE FUNCTIONS

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ABSTRACT. In the line of Duman et. al.[4], following the notion of $A^{\mathcal{I}}$-statistical convergence we study the rates of this convergence for a sequence of positive linear operators acting on the space of locally integrable functions.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers and $\mathbb{R}$ denote the set of real numbers. The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [6]. Further investigations started in this area after the pioneering works of Šalát [12] and Fridy [7]. The notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et. al. [11] as a generalization of statistical convergence using the notion of ideals. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk ([10]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [1, 2, 3, 10]). In particular, in [13, 14] the very general notion of $A^{\mathcal{I}}$-statistical convergence is introduced. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if $(i) A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (ii)$A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$. An ideal is called admissible if it satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If an ideal $\mathcal{I} \neq \{\emptyset\}$ then it is called non-trivial.
Throughout the paper $\mathcal{I}$ denotes the non-trivial admissible ideal on $\mathbb{N}$. If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{nk})_{n,k=1}^{\infty}$ is an infinite matrix, then Ax is the sequence whose n-th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

A matrix A is called regular if $A \in (c, c)$ and $\lim_{k\to\infty} A_k(x) = \lim_{k\to\infty} x_k$ for all $x = \{x_k\}_{k\in\mathbb{N}} \in c$ when c, as usual, stands for the set of all convergent sequences. The well-known Silverman-Toeplitz

2020 *Mathematics Subject Classification.* Primary: 40A35, Secondary: 47B38, 41A25, 41A36.

ISSN: Nil Volume – 1 Issue- January- June 2023

theorem states that the necessary and sufficient conditions for A to be regular are

$$
\begin{aligned}
&i) \quad ||A|| = \sup_n \sum_k |a_{nk}| < \infty; \\
&ii) \quad \lim_n a_{nk} = 0, \text{ for each } k; \\
&iii) \quad \lim_n \sum_k a_{nk} = 1.
\end{aligned}
$$

Definition 1.1. [11] *A sequence* $\{x_n\}_{n\in N}$ *in* X *is said to be* $\mathcal{I}$*-convergent to* $\xi \in X$ *if for any* $\varepsilon > 0$ *the set* $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \xi\| \geq \varepsilon\} \in \mathcal{I}$. *In this case we write* $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$.

Definition 1.2. [13, 14] *Let* $A = (a_{nk})$ *be a non-negative regular matrix. For an ideal* $\mathcal{I}$ *of* $\mathbb{N}$ *a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is said to be* $A^{\mathcal{I}}$*-statistically convergent to* L *if for any* $\varepsilon > 0$ *and* $\delta > 0$, $\left\{n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta\right\} \in \mathcal{I}$ *where* $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. *In this case, we write* $A^{\mathcal{I}}\text{-}st\text{-}\lim_n x_n = L$.

Throughout the work we will use the weight function q defined by $q(x) = 1 + x^2$ where $x \in \mathbb{R}$. We denote the space of all locally integrable functions by $L_{p,q}(\text{loc})$, that is the space of all measurable functions f for which $\left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p dt\right)^{\frac{1}{p}} \leq M_f q(x)$, $x \in \mathbb{R}$, where M_f is a positive constant depending on f and $p \geq 1$. It is known [8] that $L_{p,q}(\text{loc})$ is a normed linear space with the norm

$$
\|f\|_{p,q} := \frac{\sup_{x\in\mathbb{R}} \left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p dt\right)^{\frac{1}{p}}}{q(x)}.
$$

where $\|f\|_{p,q}$ may also depend on $h > 0$. For any real number a, b $(a < b)$, we write that $\|f; L_p(a,b)\| = (\frac{1}{b-a}\int_a^b \mid f(t) \mid^p dt)^{\frac{1}{p}}$ and $\|f; L_{p,q}(a,b)\| = \sup_{a\leq x\leq b} \frac{\|f;L_p(x-h,x+h)\|}{q(x)}$. Now the norm in $L_{p,q}(\text{loc})$ may be written in the form $\|f\|_{p,q} := \sup_{x\in\mathbb{R}} \frac{\|f;L_p(x-h,x+h)\|}{q(x)}$. For a positive linear operator T from $L_{p,q}(\text{loc})$ into $L_{p,q}(\text{loc})$, then the operator norm $\|T\|$ is given by $\|T\| := \sup_{f\neq 0} \|Tf\|_{p,q}/\|f\|_{p,q}$.

Our primary interest in this work, is to establish the rates of $A^{\mathcal{I}}$-statistical convergence of a sequence of positive linear operators defined on the space of locally integrable functions $L_{p,q}(\text{loc})$.

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2. Rates of $A^{\mathcal{I}}$-statistical convergence

We consider the following weighted modulus of continuity:

$$\omega_q(f,\delta) = \sup_{|x-y|\leq\delta} \frac{|f(y)-f(x)|}{q(x)}$$

where δ is a positive constant and $f \in L_{p,q}(\text{loc})$. It is easy to see that, for any $c > 0$ and all $f \in L_{p,q}(\text{loc})$,

$$\omega_q(f,c\delta) \leq (1+[c])\omega_q(f,\delta),$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

Definition 2.1. *Let $A = (a_{jn})$ be a non-negative regular matrix and $\{d_n\}_{n\in\mathbb{N}}$ be a positive non-increasing sequence. For an ideal $\mathcal{I}$ of $\mathbb{N}$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L with the rate of $o(d_n)$ if for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ j \in \mathbb{N} : \frac{1}{d_j} \sum_{n\in K(\varepsilon)} a_{jn} \geq \delta \right\} \in \mathcal{I}$$

where $K(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. In this case, we write $A^{\mathcal{I}}\text{-}st\text{-}o(d_n)\text{-}\lim_n x_n = L$.

In order to give our main result we need the following lemma which is proven in [4].

Lemma 2.1. *Let $\{T_n\}$ be a sequence of positive linear operators acting from $L_{p,q}(loc)$ into $L_{p,q}(loc)$. Then for each $n \in \mathbb{N}$ and $\delta > 0$, and for every function f that is continuous and bounded on the whole real axis, we have*

$$\begin{aligned} \|T_n f - f; L_{p,q}(a,b)\| \leq\ & C_1\omega_q(f,\delta)\|T_n f_0 - f_0\|_{p,q} + C_1\omega_q(f,\delta) + \frac{C_1}{\delta^2}\omega_q(f,\delta)\|T_n\varphi_x\|_{p,q} \\ & + C_2\|T_n f_0 - f_0\|_{p,q} \end{aligned}$$

where, $f_0(t) := 1, \varphi_x(t) := (t-x)^2, C_1 := \sup_{a\leq x\leq b} q(x), C_2 := \sup_{a\leq x\leq b} |f(x)|$.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be positive non-increasing sequences. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into $L_{p,q}(loc)$ and $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$. Assume that, for each continuous and bounded function f on the real line, the following conditions hold:*

$$(i) A^{\mathcal{I}}\text{-}st\text{-}o(a_n)\text{-}\lim_n \|T_n f_0 - f_0\|_{p,q} = 0;$$

$$(ii) A^{\mathcal{I}}\text{-}st\text{-}o(b_n)\text{-}\lim_n \omega_q(f,\alpha_n) = 0,$$

with $\alpha_n = \sqrt{\|T_n\varphi_x\|_{p,q}}$.
Then we have

$$A^{\mathcal{I}}\text{-}st\text{-}o(c_n)\text{-}\lim_n \|T_n f - f; L_{p,q}(a,b)\| = 0,$$

where $c_n := \max\{a_n, b_n\}$. Similar results hold when little "o" is replaced by big "O".

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ISSN: Nil Volume – 1 Issue- January- June 2023

Proof. Let $\delta = \alpha_n = \sqrt{\|T_n\varphi_x\|_{p,q}}$. Then from Lemma 2.1 for every $n \in \mathbb{N}$, we get that

$$\begin{aligned}\|T_nf - f; L_{p,q}(a,b)\| &\leq C_1\omega_q(f,\alpha_n)\|T_nf_0 - f_0\|_{p,q} + C_1\omega_q(f,\alpha_n) + \frac{C_1}{\delta^2}\omega_q(f,\alpha_n)\|T_n\varphi_x\|_{p,q} \\ &\quad + C_2\|T_nf_0 - f_0\|_{p,q} \\ &\leq C_1\omega_q(f,\alpha_n)\|T_nf_0 - f_0\|_{p,q} + 2C_1\omega_q(f,\alpha_n) + C_2\|T_nf_0 - f_0\|_{p,q}.\end{aligned}$$

For any $\varepsilon > 0$, let us define the following sets

$$D := \{n \in \mathbb{N} : \|T_nf - f; L_{p,q}(a,b)\| \geq \varepsilon\};$$

$$D_1 := \{n \in \mathbb{N} : \omega_q(f,\alpha_n)\|T_nf_0 - f_0\|_{p,q} \geq \frac{\varepsilon}{3C_1}\};$$

$$D_2 := \{n \in \mathbb{N} : \omega_q(f,\alpha_n) \geq \frac{\varepsilon}{6C_1}\};$$

$$D_3 := \{n \in \mathbb{N} : \|T_nf_0 - f_0\|_{p,q} \geq \frac{\varepsilon}{3C_2}\}.$$

Then it follows that $D \subseteq D_1 \cup D_2 \cup D_3$. Also if we define the sets

$$D_1' := \{n \in \mathbb{N} : \omega_q(f,\alpha_n) \geq \sqrt{\frac{\varepsilon}{3C_1}}\};$$

$$D_1'' := \{n \in \mathbb{N} : \|T_nf_0 - f_0\|_{p,q} \geq \sqrt{\frac{\varepsilon}{3C_1}}\}$$

then we can observe that $D \subseteq D_1' \cup D_1'' \cup D_2 \cup D_3$. Now since $c_j = \max\{a_j, b_j\}$, we get, for every $j \in \mathbb{N}$, that

$$\frac{1}{c_j}\sum_{n\in D} a_{jn} \leq \frac{1}{b_j}\sum_{n\in D_1'} a_{jn} + \frac{1}{a_j}\sum_{n\in D_1''} a_{jn} + \frac{1}{b_j}\sum_{n\in D_2} a_{jn} + \frac{1}{a_j}\sum_{n\in D_3} a_{jn}$$

which implies that for any $\sigma > 0$,

$$\begin{aligned}\{j \in \mathbb{N} : \frac{1}{c_j}\sum_{n\in D} a_{jn} \geq \sigma\} &\subseteq \{j \in \mathbb{N} : \frac{1}{b_j}\sum_{n\in D_1'} a_{jn} \geq \frac{\sigma}{4}\} \cup \{j \in \mathbb{N} : \frac{1}{a_j}\sum_{n\in D_1''} a_{jn} \geq \frac{\sigma}{4}\} \\ &\quad \cup \{j \in \mathbb{N} : \frac{1}{b_j}\sum_{n\in D_2} a_{jn} \geq \frac{\sigma}{4}\} \cup \{j \in \mathbb{N} : \frac{1}{a_j}\sum_{n\in D_3} a_{jn} \geq \frac{\sigma}{4}\}.\end{aligned}$$

Therefore from the hypothesis $\{j \in \mathbb{N} : \frac{1}{c_j}\sum_{j\in D} a_{jn} \geq \sigma\} \in \mathcal{I}$ for any $\sigma > 0$. Hence

$$A^{\mathcal{I}}\text{-}st\text{-}o(c_n)\text{-}\lim_n \|T_nf - f; L_{p,q}(a,b)\| = 0$$

ISSN: Nil Volume – 1 Issue- January- June 2023

3. Conclusion

If we replace the ideal $\mathcal{I}$ by the ideal $\mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, Theorem 2.1 of this paper implies Theorem 2.2 in [4]. Furthermore, if we replace the matrix $A = (a_{jn})$ by the identity matrix and take $\mathcal{I} = \mathcal{I}_{fin}$, then Theorem 2.1 immediately gives the ordinary rates of convergence.

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