

## A NOTE ON RATES OF $A^{\mathcal{I}}$ -STATISTICAL CONVERGENCE OF OPERATORS IN THE SPACE OF LOCALLY INTEGRABLE FUNCTIONS

DR. SUDIPTA DUTTA  
ASSISTANT PROFESSOR  
DEPARTMENT OF MATHEMATICS  
GOVT. GENERAL DEGREE COLLEGE AT MANBAZAR-II  
PURULIA, PIN-723131, WEST BENGAL, INDIA  
EMAIL:DRSUDIPTA.PROF@GMAIL.COM

**ABSTRACT.** In the line of Duman et. al.[4], following the notion of  $A^{\mathcal{I}}$ -statistical convergence we study the rates of this convergence for a sequence of positive linear operators acting on the space of locally integrable functions.

### 1. Introduction and Background

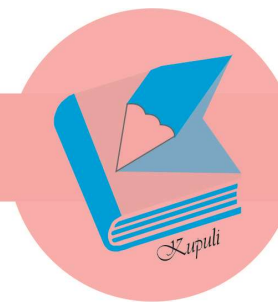
Throughout the paper  $\mathbb{N}$  will denote the set of all positive integers and  $\mathbb{R}$  denote the set of real numbers. The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [6]. Further investigations started in this area after the pioneering works of Šalát [12] and Fridy [7]. The notion of  $\mathcal{I}$ -convergence of real sequences was introduced by Kostyrko et. al. [11] as a generalization of statistical convergence using the notion of ideals. On the other hand statistical convergence was generalized to  $A$ -statistical convergence by Kolk ([10]). Later a lot of works have been done on matrix summability and  $A$ -statistical convergence (see [1, 2, 3, 10]). In particular, in [13, 14] the very general notion of  $A^{\mathcal{I}}$ -statistical convergence is introduced. A family  $\mathcal{I} \subset 2^Y$  of subsets of a non-empty set  $Y$  is said to be an ideal in  $Y$  if (i)  $A, B \in \mathcal{I}$  implies  $A \cup B \in \mathcal{I}$  (ii)  $A \in \mathcal{I}, B \subset A$  implies  $B \in \mathcal{I}$ . An ideal is called admissible if it satisfies  $\{x\} \in \mathcal{I}$  for each  $x \in Y$ . If an ideal  $\mathcal{I} \neq \{\emptyset\}$  then it is called non-trivial. Throughout the paper  $\mathcal{I}$  denotes the non-trivial admissible ideal on  $\mathbb{N}$ . If  $\{x_k\}_{k \in \mathbb{N}}$  is a sequence of real numbers and  $A = (a_{nk})_{n,k=1}^{\infty}$  is an infinite matrix, then  $Ax$  is the sequence whose  $n$ -th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} \cdot x_k.$$

A matrix  $A$  is called regular if  $A \in (c, c)$  and  $\lim_{k \rightarrow \infty} A_k(x) = \lim_{k \rightarrow \infty} x_k$  for all  $x = \{x_k\}_{k \in \mathbb{N}} \in c$  when  $c$ , as usual, stands for the set of all convergent sequences. The well-known Silverman-Toeplitz

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theorem states that the necessary and sufficient conditions for  $A$  to be regular are

- i)  $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$ ;
- ii)  $\lim_n a_{nk} = 0$ , for each  $k$ ;
- iii)  $\lim_n \sum_k a_{nk} = 1$ .

**Definition 1.1.** [11] A sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  is said to be  $\mathcal{I}$ -convergent to  $\xi \in X$  if for any  $\varepsilon > 0$  the set  $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - \xi\| \geq \varepsilon\} \in \mathcal{I}$ . In this case we write  $\mathcal{I} - \lim_{n \rightarrow \infty} x_n = \xi$ .

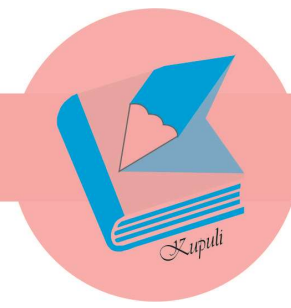
**Definition 1.2.** [13, 14] Let  $A = (a_{nk})$  be a non-negative regular matrix. For an ideal  $\mathcal{I}$  of  $\mathbb{N}$  a sequence  $\{x_n\}_{n \in \mathbb{N}}$  is said to be  $A^{\mathcal{I}}$ -statistically convergent to  $L$  if for any  $\varepsilon > 0$  and  $\delta > 0$ ,  $\left\{ n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I}$  where  $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ . In this case, we write  $A^{\mathcal{I}}\text{-st-}\lim_n x_n = L$ .

Throughout the work we will use the weight function  $q$  defined by  $q(x) = 1 + x^2$  where  $x \in \mathbb{R}$ . We denote the space of all locally integrable functions by  $L_{p,q}(\text{loc})$ , that is the space of all measurable functions  $f$  for which  $\left( \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right)^{\frac{1}{p}} \leq M_f q(x)$ ,  $x \in \mathbb{R}$ , where  $M_f$  is a positive constant depending on  $f$  and  $p \geq 1$ . It is known [8] that  $L_{p,q}(\text{loc})$  is a normed linear space with the norm

$$\|f\|_{p,q} := \frac{\sup_{x \in \mathbb{R}} \left( \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p dt \right)^{\frac{1}{p}}}{q(x)}.$$

where  $\|f\|_{p,q}$  may also depend on  $h > 0$ . For any real number  $a, b$  ( $a < b$ ), we write that  $\|f; L_p(a, b)\| = \left( \frac{1}{b-a} \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$  and  $\|f; L_{p,q}(a, b)\| = \sup_{a \leq x \leq b} \frac{\|f; L_p(x-h, x+h)\|}{q(x)}$ . Now the norm in  $L_{p,q}(\text{loc})$  may be written in the form  $\|f\|_{p,q} := \sup_{x \in \mathbb{R}} \frac{\|f; L_p(x-h, x+h)\|}{q(x)}$ . For a positive linear operator  $T$  from  $L_{p,q}(\text{loc})$  into  $L_{p,q}(\text{loc})$ , then the operator norm  $\|T\|$  is given by  $\|T\| := \sup_{f \neq 0} \|Tf\|_{p,q} / \|f\|_{p,q}$ .

Our primary interest in this work, is to establish the rates of  $A^{\mathcal{I}}$ -statistical convergence of a sequence of positive linear operators defined on the space of locally integrable functions  $L_{p,q}(\text{loc})$ .



2. Rates of  $A^{\mathcal{I}}$ -statistical convergence

We consider the following weighted modulus of continuity:

$$\omega_q(f, \delta) = \sup_{|x-y| \leq \delta} |f(y) - f(x)| q(x)$$

where  $\delta$  is a positive constant and  $f \in L_{p,q}(loc)$ . It is easy to see that, for any  $c > 0$  and all  $f \in L_{p,q}(loc)$ ,

$$\omega_q(f, c\delta) \leq (1 + [c])\omega_q(f, \delta),$$

where  $[c]$  is defined to be the greatest integer less than or equal to  $c$ .

**Definition 2.1.** Let  $A = (a_{jn})$  be a non-negative regular matrix and  $\{d_n\}_{n \in \mathbb{N}}$  be a positive non-increasing sequence. For an ideal  $\mathcal{I}$  of  $\mathbb{N}$ , a sequence  $\{x_n\}_{n \in \mathbb{N}}$  is said to be  $A^{\mathcal{I}}$ -statistically convergent to  $L$  with the rate of  $o(d_n)$  if for any  $\varepsilon > 0$  and  $\delta > 0$ ,

$$\left\{ j \in \mathbb{N} : \frac{1}{d_j} \sum_{n \in K(\varepsilon)} a_{jn} \geq \delta \right\} \in \mathcal{I}$$

where  $K(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ . In this case, we write  $A^{\mathcal{I}}$ -st- $o(d_n)$ - $\lim_n x_n = L$ .

In order to give our main result we need the following lemma which is proven in [4].

**Lemma 2.1.** Let  $\{T_n\}$  be a sequence of positive linear operators acting from  $L_{p,q}(loc)$  into  $L_{p,q}(loc)$ . Then for each  $n \in \mathbb{N}$  and  $\delta > 0$ , and for every function  $f$  that is continuous and bounded on the whole real axis, we have

$$\begin{aligned} \|T_n f - f; L_{p,q}(a, b)\| &\leq C_1 \omega_q(f, \delta) \|T_n f_0 - f_0\|_{p,q} + C_1 \omega_q(f, \delta) + \frac{C_1}{\delta^2} \omega_q(f, \delta) \|T_n \varphi_x\|_{p,q} \\ &\quad + C_2 \|T_n f_0 - f_0\|_{p,q} \end{aligned}$$

where,  $f_0(t) := 1, \varphi_x(t) := (t - x)^2, C_1 := \sup_{a \leq x \leq b} q(x), C_2 := \sup_{a \leq x \leq b} |f(x)|$ .

**Theorem 2.1.** Let  $A = (a_{jn})$  be a non-negative regular summability matrix and let  $\{a_n\}_{n \in \mathbb{N}}$  and  $\{b_n\}_{n \in \mathbb{N}}$  be positive non-increasing sequences. Let  $\{T_n\}$  be a sequence of positive linear operators from  $L_{p,q}(loc)$  into  $L_{p,q}(loc)$  and  $\mathcal{I}$  is an admissible ideal on  $\mathbb{N}$ . Assume that, for each continuous and bounded function  $f$  on the real line, the following conditions hold:

- (i)  $A^{\mathcal{I}}$ -st- $o(a_n)$ - $\lim_n \|T_n f_0 - f_0\|_{p,q} = 0$ ;
- (ii)  $A^{\mathcal{I}}$ -st- $o(b_n)$ - $\lim_n \omega_q(f, \alpha_n) = 0$ ,

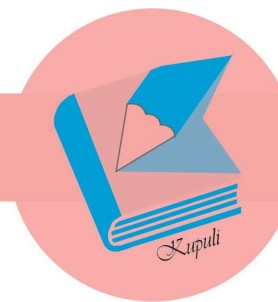
with  $\alpha_n = \sqrt{\|T_n \varphi_x\|_{p,q}}$ .

Then we have

$$A^{\mathcal{I}}$$
-st- $o(c_n)$ - $\lim_n \|T_n f - f; L_{p,q}(a, b)\| = 0,$

where  $c_n := \max\{a_n, b_n\}$ . Similar results hold when little “o” is replaced by big “O”.





*Proof.* Let  $\delta = \alpha_n = \sqrt{\|T_n \varphi_x\|_{p,q}}$ . Then from Lemma 2.1 for every  $n \in \mathbb{N}$ , we get that

$$\begin{aligned} \|T_n f - f; L_{p,q}(a, b)\| &\leq C_1 \omega_q(f, \alpha_n) \|T_n f_0 - f_0\|_{p,q} + C_1 \omega_q(f, \alpha_n) + \frac{C_1}{\delta^2} \omega_q(f, \alpha_n) \|T_n \varphi_x\|_{p,q} \\ &\quad + C_2 \|T_n f_0 - f_0\|_{p,q} \\ &\leq C_1 \omega_q(f, \alpha_n) \|T_n f_0 - f_0\|_{p,q} + 2C_1 \omega_q(f, \alpha_n) + C_2 \|T_n f_0 - f_0\|_{p,q}. \end{aligned}$$

For any  $\varepsilon > 0$ , let us define the following sets

$$D := \{n \in \mathbb{N} : \|T_n f - f; L_{p,q}(a, b)\| \geq \varepsilon\};$$

$$D_1 := \{n \in \mathbb{N} : \omega_q(f, \alpha_n) \|T_n f_0 - f_0\|_{p,q} \geq \frac{\varepsilon}{3C_1}\};$$

$$D_2 := \{n \in \mathbb{N} : \omega_q(f, \alpha_n) \geq \frac{\varepsilon}{6C_1}\};$$

$$D_3 := \{n \in \mathbb{N} : \|T_n f_0 - f_0\|_{p,q} \geq \frac{\varepsilon}{3C_2}\}.$$

Then it follows that  $D \subseteq D_1 \cup D_2 \cup D_3$ . Also if we define the sets

$$D'_1 := \{n \in \mathbb{N} : \omega_q(f, \alpha_n) \geq \sqrt{\frac{\varepsilon}{3C_1}}\};$$

$$D''_1 := \{n \in \mathbb{N} : \|T_n f_0 - f_0\|_{p,q} \geq \sqrt{\frac{\varepsilon}{3C_1}}\}$$

then we can observe that  $D \subseteq D'_1 \cup D''_1 \cup D_2 \cup D_3$ . Now since  $c_j = \max\{a_j, b_j\}$ , we get, for every  $j \in \mathbb{N}$ , that

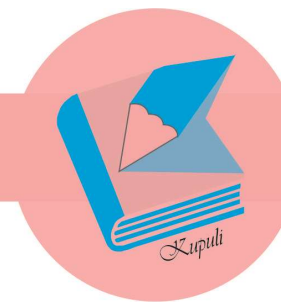
$$\frac{1}{c_j} \sum_{n \in D} a_{jn} \leq \frac{1}{b_j} \sum_{n \in D'_1} a_{jn} + \frac{1}{a_j} \sum_{n \in D''_1} a_{jn} + \frac{1}{b_j} \sum_{n \in D_2} a_{jn} + \frac{1}{a_j} \sum_{n \in D_3} a_{jn}$$

which implies that for any  $\sigma > 0$ ,

$$\begin{aligned} \{j \in \mathbb{N} : \frac{1}{c_j} \sum_{n \in D} a_{jn} \geq \sigma\} &\subseteq \{j \in \mathbb{N} : \frac{1}{b_j} \sum_{n \in D'_1} a_{jn} \geq \frac{\sigma}{4}\} \cup \{j \in \mathbb{N} : \frac{1}{a_j} \sum_{n \in D''_1} a_{jn} \geq \frac{\sigma}{4}\} \\ &\quad \cup \{j \in \mathbb{N} : \frac{1}{b_j} \sum_{n \in D_2} a_{jn} \geq \frac{\sigma}{4}\} \cup \{j \in \mathbb{N} : \frac{1}{a_j} \sum_{n \in D_3} a_{jn} \geq \frac{\sigma}{4}\}. \end{aligned}$$

Therefore from the hypothesis  $\{j \in \mathbb{N} : \frac{1}{c_j} \sum_{j \in D} a_{jn} \geq \sigma\} \in \mathcal{I}$  for any  $\sigma > 0$ . Hence

$$A^I\text{-st-}o(c_n)\text{-}\lim_n \|T_n f - f; L_{p,q}(a, b)\| = 0$$



### 3. CONCLUSION

If we replace the ideal  $\mathcal{I}$  by the ideal  $\mathcal{I}_{fin}$ , the ideal of all finite subsets of  $\mathbb{N}$ , Theorem 2.1 of this paper implies Theorem 2.2 in [4]. Furthermore, if we replace the matrix  $A = (a_{jn})$  by the identity matrix and take  $\mathcal{I} = \mathcal{I}_{fin}$ , then Theorem 2.1 immediately gives the ordinary rates of convergence.

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